Theory of force-free electromagnetic fields. II. Configuration with symmetry

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We develop a method for applying the general theory of the force-free electromagnetic field given in the preceding paper to configurations with some symmetry. The electromagnetic field configuration invariant under the action generated by one Killing vector and the configuration invariant under the action generated by two Killing vectors are studied. The Euler potentials have specific forms when the electromagnetic field has symmetry. General forms of the Euler potentials in these two cases are determined by the symmetry assumed. As an example of the configuration with one invariant direction, the time-dependent axisymmetric configuration is studied. The example of the configuration with two invariant directions is the stationary and axisymmetric configuration. The relation between the present theory and the traditional way to treat the stationary and axisymmetric configuration is clarified. Lastly, using Noether's identities, we clarify the relation between the geometrical properties of the conserved fluxes and the symmetry of the configuration. [S1063-651X(97)03108-5]

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I. INTRODUCTION

In the preceding paper (henceforth referred to as paper I), we have presented a general theory to deal with the force-free electromagnetic field. In that paper, we have treated the force-free electromagnetic field by a field theoretical method that is free from the symmetry of the field configuration. Indeed, the force-free electromagnetic field was described as a nonlinear scalar field theory for two Euler potentials. Since the motivation of that paper is formulation of a general theory, a viewpoint that does not depend on any symmetry of the configuration was stressed throughout. In many astronomical applications, however, we treat field configurations with some symmetry. Although real astronomical bodies do not have any exact symmetry, of course, some approximate symmetry exists in many cases. Generally, the condition for the symmetry greatly simplifies the mathematical analysis. Therefore development of a systematic method for applying the general theory of the force-free electromagnetic field to configurations with some symmetry is necessarily a next step of our investigation.

As shown in paper I, a force-free electromagnetic field $F_{\mu\nu}$, more generally a degenerate electromagnetic field, is written by two Euler potentials ϕ_1 and ϕ_2 as

$$F_{\mu\nu}=\partial_\mu\phi_1\partial_\nu\phi_2-\partial_\mu\phi_2\partial_\nu\phi_1. \tag{1.1}$$

A two-dimensional surface on which ϕ_1 and ϕ_2 are constant is called the flux surface. A tangent vector field of the flux surface is called the generator of the flux surface. The theory of the force-free electromagnetic field is a geometric theory largely based on the existence of the flux surface. The flux surfaces introduce the magnetic field lines on a given three-space. Namely, the magnetic field line is an intersection between a flux surface and the three-space. An expression of the vector potential that yields the electromagnetic field (1.1) is given by

$$A_\mu=\frac{1}{2}(\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1). \tag{1.2}$$

Of course, the vector potential cannot be determined uniquely. The arbitrariness in the vector potential is clarified in paper I.

Combining the force-free equation $F_{\mu\nu}J^\nu=0$ [1] with a Maxwell equation, we have

$$\partial_\nu\phi_1\partial_\lambda\{\sqrt{-g}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\}=0,$$
$$\partial_\nu\phi_2\partial_\lambda\{\sqrt{-g}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\}=0. \tag{1.3}$$

This is the basic equation of the force-free electromagnetic approximation. The basic equation is derived from an action principle. The Lagrangian scalar L that yields Eq. (1.3) is

$$L=-\frac{1}{16\pi}(\partial^\nu\phi_1\partial^\lambda\phi_2-\partial^\nu\phi_2\partial^\lambda\phi_1)\times(\partial_\nu\phi_1\partial_\lambda\phi_2-\partial_\nu\phi_2\partial_\lambda\phi_1). \tag{1.4}$$

In this work we will develop a method for applying these equations systematically to the configurations with some symmetry. In configurations with symmetry, the Euler potentials must have specific functional forms so as to embody the symmetry in the electromagnetic field. We can decide this specific form indeed by the given requirements for the symmetry. Further, as we will see, this formulation enables us to see the relation between the symmetry of the configuration and the properties of the conserved flux by means of Noether's identities.

The plan of this paper is as follows. Section II treats configurations that have one invariant direction. The forms of the Euler potentials are determined. As an example, the time-dependent axisymmetric case is examined. In Sec. III we treat configurations that have two invariant directions. We show that generally two cases may arise. The forms of the Euler potentials are determined in each of the two cases. As an example, the stationary and axisymmetric configuration is considered. Further, the relation between the traditional treatment of the stationary and axisymmetric configuration and our theory is clarified. In Sec. IV relations between the geometrical properties of the conserved flux and the symmetry of the configuration are discussed making use of Noether's identities.

The metric signature $(-+++)$ and the units in which $c=G=1$ are used as in paper I.

II. CONFIGURATION WITH ONE INVARIANT DIRECTION

A. Restriction on the Euler potentials

Let us begin with the degenerate electromagnetic field invariant under the action generated by one Killing vector. Let ζ^μ denote the Killing vector field. We assume that the degenerate electromagnetic field satisfies

$$\pounds_\zeta F_{\mu\nu}=0, \tag{2.1}$$

where $\pounds_\zeta$ is the Lie derivative with respect to ζ^μ. We look for the general forms of the Euler potentials that yield the force-free electromagnetic field satisfying Eq. (2.1). Here, we derive the conditions for the Euler potentials by a heuristic argument first and later discuss generality of the conditions.

At first sight, it seems that the condition $\pounds_\zeta \phi_i=0$ $(i=1,2)$ is adequate. Of course, this yields an electromagnetic field satisfying condition (2.1). However, this is not a general condition. In fact, this requirement makes the electromagnetic field satisfy $F_{\mu\nu}\zeta^\nu=0$. This implies that three of six components of the electromagnetic field tensor vanish identically. For example, if ζ^μ is the stationary Killing vector $\partial_t=\mathbf{k}$, then $F_{ti}=0$ $(i=1\text{–}3)$ follows. This implies that the electric field vanishes identically. Thus, if $\pounds_{\mathbf{k}}\phi_i=0 (i=1,2)$ is a general condition for $\pounds_{\mathbf{k}}F_{\mu\nu}=0$, we must conclude that the stationary degenerate electromagnetic field cannot have the electric field at all. Evidently, this is a too restrictive condition. Therefore we examine the consequence of the condition

$$\pounds_\zeta \phi_1=0. \tag{2.2}$$

Lie differentiating $F_{\mu\nu}$ with respect to ζ^μ under condition (2.2), we have

$$\pounds_\zeta F_{\mu\nu}=\partial_\mu\phi_1\partial_\nu(\pounds_\zeta\phi_2)-\partial_\mu(\pounds_\zeta\phi_2)\partial_\nu\phi_1. \tag{2.3}$$

Obviously, we have $\pounds_\zeta F_{\mu\nu}=0$ if ϕ_2 satisfies

$$\pounds_\zeta\phi_2=\zeta^\mu\partial_\mu\phi_2=f(\phi_1), \tag{2.4}$$

where $f(\phi_1)$ is an arbitrary function of ϕ_1. Thus not $\pounds_\zeta\phi_2=0$ but condition (2.4) is sufficient for Eq. (2.1), if Eq. (2.2) is assumed.

Equations (2.2) and (2.4) seem to be general conditions for the Euler potentials. Further, $f(\phi_1)$ appears like an integral on the three-dimensional surface of constant ϕ_1. However, there is room for further simplification. Namely, we can set $f(\phi_1)=1$ without any loss of generality except for the case in which $f(\phi_1)$ vanishes identically. In fact, as shown in paper I, two sets of the Euler potentials (ϕ_1,ϕ_2) and $(\tilde\phi_1,\tilde\phi_2)$ give the same electromagnetic field if they relate as

$$\tilde\phi_1=\tilde\phi_1(\phi_1,\phi_2),\quad \tilde\phi_2=\tilde\phi_2(\phi_1,\phi_2), \tag{2.5}$$

with

$$\frac{\partial\tilde\phi_1}{\partial\phi_1}\frac{\partial\tilde\phi_2}{\partial\phi_2}-\frac{\partial\tilde\phi_1}{\partial\phi_2}\frac{\partial\tilde\phi_2}{\partial\phi_1}=\frac{\partial(\tilde\phi_1,\tilde\phi_1)}{\partial(\phi_1,\phi_2)}=1. \tag{2.6}$$

The Euler potentials are determined within this arbitrariness. Making use of this arbitrariness, we can introduce a new set of the Euler potentials $\tilde\phi_1$ and $\tilde\phi_2$ as

$$\tilde\phi_1=\int f(\phi_1)d\phi_1,\quad \tilde\phi_2=\frac{1}{f(\phi_1)}\phi_2, \tag{2.7}$$

from the Euler potentials satisfying Eqs. (2.2) and (2.4). The exception is the case in which $f(\phi_1)$ is identically zero. Evidently, Eqs. (2.7) are an example of the transformation defined by Eqs. (2.5) and (2.6). In fact, it is easy to see that this transformation does not change the electromagnetic field. By equations (2.2) and (2.4), $\tilde\phi_2$ satisfies $\pounds_\zeta\tilde\phi_2=1$. Of course, $\pounds_\zeta\tilde\phi_1=0$ is still satisfied. Therefore the Euler potentials that yield a degenerate electromagnetic field invariant under the action generated by ζ satisfy

$$\pounds_\zeta\phi_1=0,\quad \pounds_\zeta\phi_2=1, \tag{2.8}$$

except for the special case

$$\pounds_\zeta\phi_1=0,\quad \pounds_\zeta\phi_2=0. \tag{2.9}$$

The latter corresponds to the $F_{\mu\nu}\zeta^\nu=0$ case mentioned above. Equations (2.8) and (2.9) are differential equations for the Euler potentials. Thus once the Killing vector ζ is given explicitly, we can decide the forms of the Euler potentials solving these equations. We will illustrate this procedure in the next subsection.

Next, let us show that condition (2.8) or condition (2.9) is in fact a general condition. From Eq. (2.1), we have

$$\partial_\mu(F_{\nu\lambda}\zeta^\lambda)-\partial_\nu(F_{\mu\lambda}\zeta^\lambda)=0. \tag{2.10}$$

This implies that $F_{\mu\lambda}\zeta^\lambda$ is written as $F_{\mu\lambda}\zeta^\lambda=\partial_\mu f$ by a function f. Let ξ^μ be a generator of the flux surface. Namely, ξ^μ is a vector field satisfying $F_{\mu\nu}\xi^\nu=0$. Then we have

$$\xi^\mu F_{\mu\lambda}\zeta^\lambda=\xi^\mu\partial_\mu f=0. \tag{2.11}$$

This means that $\partial_\mu f$ is orthogonal to the flux surface. Thus f is a function of the Euler potentials. Thus we have

$$F_{\mu\lambda}\zeta^\lambda=\partial_\mu f(\phi_1,\phi_2), \tag{2.12}$$

where $f(\phi_1,\phi_2)$ is a function of ϕ_1 and ϕ_2. Substituting Eq. (1.1) into $F_{\mu\nu}$ in the above equation, we have

$$\partial_\mu\phi_1(\zeta^\nu\partial_\nu\phi_2)-\partial_\mu\phi_2(\zeta^\nu\partial_\nu\phi_1)=\frac{\partial f}{\partial\phi_1}\partial_\mu\phi_1+\frac{\partial f}{\partial\phi_2}\partial_\mu\phi_2. \tag{2.13}$$

So long as $F_{\mu\nu}$ does not vanish identically, $\partial_\mu\phi_1$ and $\partial_\mu\phi_2$ are independent. Comparing both sides of the above equation, we find the relations

$$\pounds_\zeta\phi_1=\zeta^\mu\partial_\mu\phi_1=-\frac{\partial f}{\partial\phi_2},\quad \pounds_\zeta\phi_2=\zeta^\mu\partial_\mu\phi_2=\frac{\partial f}{\partial\phi_1}. \tag{2.14}$$

Here we should note that Eqs. (2.14) are transformed ''covariantly'' under the transformation of the Euler potentials defined by Eqs. (2.5) and (2.6). Namely, the same relations as Eqs. (2.14), but in which ϕ_i is replaced by $\tilde{\phi}_i$, are satisfied by any set of $(\tilde{\phi}_1,\tilde{\phi}_2)$ if $\tilde{\phi}_i$ relates to ϕ_i as Eqs. (2.5) and (2.6). A proof is as follows. Regarding $\tilde{\phi}_1$ and $\tilde{\phi}_2$ as functions of ϕ_1 and ϕ_2, we have

$$\zeta^\mu\partial_\mu\tilde{\phi}_1=\frac{\partial\tilde{\phi}_1}{\partial\phi_1}\zeta^\mu\partial_\mu\phi_1+\frac{\partial\tilde{\phi}_1}{\partial\phi_2}\zeta^\mu\partial_\mu\phi_2,$$

$$\zeta^\mu\partial_\mu\tilde{\phi}_2=\frac{\partial\tilde{\phi}_2}{\partial\phi_1}\zeta^\mu\partial_\mu\phi_1+\frac{\partial\tilde{\phi}_2}{\partial\phi_2}\zeta^\mu\partial_\mu\phi_2, \tag{2.15}$$

and

$$\frac{\partial f}{\partial\phi_1}=\frac{\partial\tilde{\phi}_1}{\partial\phi_1}\frac{\partial f}{\partial\tilde{\phi}_1}+\frac{\partial\tilde{\phi}_2}{\partial\phi_1}\frac{\partial f}{\partial\tilde{\phi}_2},$$

$$\frac{\partial f}{\partial\phi_2}=\frac{\partial\tilde{\phi}_1}{\partial\phi_2}\frac{\partial f}{\partial\tilde{\phi}_1}+\frac{\partial\tilde{\phi}_2}{\partial\phi_2}\frac{\partial f}{\partial\tilde{\phi}_2}. \tag{2.16}$$

We substitute Eqs. (2.16) into Eqs. (2.14). Further, we replace $\zeta^\mu\partial_\mu\phi_i(i=1,2)$ in Eqs. (2.15) by Eqs. (2.14). Then together with $\partial(\tilde{\phi}_1,\tilde{\phi}_1)/\partial(\phi_1,\phi_2)=\partial(\tilde{\phi}_2,\tilde{\phi}_2)/\partial(\phi_1,\phi_2)=0$, we have

$$\zeta^\mu\partial_\mu\tilde{\phi}_1=-\frac{\partial(\tilde{\phi}_1,\tilde{\phi}_2)}{\partial(\phi_1,\phi_2)}\frac{\partial f}{\partial\tilde{\phi}_2},\quad \zeta^\mu\partial_\mu\tilde{\phi}_2=\frac{\partial(\tilde{\phi}_1,\tilde{\phi}_2)}{\partial(\phi_1,\phi_2)}\frac{\partial f}{\partial\tilde{\phi}_1}. \tag{2.17}$$

Thus $\partial(\tilde{\phi}_1,\tilde{\phi}_2)/\partial(\phi_1,\phi_2)=1$ guarantees

$$\zeta^\mu\partial_\mu\tilde{\phi}_1=-\frac{\partial f}{\partial\tilde{\phi}_2},\quad \zeta^\mu\partial_\mu\tilde{\phi}_2=\frac{\partial f}{\partial\tilde{\phi}_1}. \tag{2.18}$$

Therefore we have the same form of equation as Eq. (2.16) for $\tilde{\phi}_1$ and $\tilde{\phi}_2$.

Using this property, we can find a new set of the Euler potentials that transforms Eqs. (2.14) to the form of Eqs. (2.8) or Eq. (2.11). The following two cases should be considered separately.

(i) When $\partial f/\partial\phi_1=0$, f becomes a function of ϕ_2, i.e., $f=f(\phi_2)$. Thus Eqs. (2.14) become

$$\zeta^\mu\partial_\mu\phi_2=0,\quad \zeta^\mu\partial_\mu\phi_1=-\frac{df}{d\phi_2}=\kappa(\phi_2). \tag{2.19}$$

This is equivalent to Eq. (2.4) except for the point that ϕ_1 and ϕ_2 are exchanged. Thus we can construct the Euler potentials satisfying Eq. (2.8).

(ii) When $\partial f/\partial\phi_1\neq0$, we can find a transformation that satisfies $\tilde{\phi}_1(\phi_1,\phi_2)=f(\phi_1,\phi_2)$ in addition to conditions (2.5) and (2.6). In fact, by virtue of $\partial f/\partial\phi_1\neq0$, we can solve $\tilde{\phi}_1=f(\phi_1,\phi_2)$ for ϕ_1. Thus we have

$$d\phi_1\wedge d\phi_2=\frac{\partial\phi_1}{\partial\tilde{\phi}_1}d\tilde{\phi}_1\wedge d\phi_2. \tag{2.20}$$

Defining $\tilde{\phi}_2$ as

$$\tilde{\phi}_2=\int\frac{\partial\phi_1(\tilde{\phi}_1,\phi_2)}{\partial\tilde{\phi}_1}d\phi_2, \tag{2.21}$$

thus we arrive at $d\phi_1\wedge d\phi_2=d\tilde{\phi}_1\wedge d\tilde{\phi}_2$, i.e., $\partial(\tilde{\phi}_1,\tilde{\phi}_2)/\partial(\phi_1,\phi_2)=1$. Since the definition of $\tilde{\phi}_1$ implies $f(\tilde{\phi}_1,\tilde{\phi}_2)=f(\phi_1(\tilde{\phi}_1,\tilde{\phi}_2),\phi_2(\tilde{\phi}_1,\tilde{\phi}_2))=\tilde{\phi}_1$, we have

$$\zeta^\mu\partial_\mu\tilde{\phi}_1=-\frac{\partial f}{\partial\tilde{\phi}_2}=0,\quad \zeta^\mu\partial_\mu\tilde{\phi}_2=\frac{\partial f}{\partial\tilde{\phi}_1}=1, \tag{2.22}$$

by the ''covariance'' of Eq. (2.14). Thus $\tilde{\phi}_1$ and $\tilde{\phi}_2$ satisfy the same condition as Eq. (2.8). Further, Eq. (2.9) is a special case of (i) in which $\kappa(\phi_2)=0$, because Eq. (2.9) implies $\zeta^\mu\partial_\mu\tilde{\phi}_1=\zeta^\mu\partial_\mu\tilde{\phi}_2=0$. From (i) and (ii) therefore we can conclude that it is always possible to find a set of the Euler potentials satisfying Eqs. (2.8) or Eq. (2.11) when the degenerate electromagnetic field is invariant under the action generated by one Killing vector. The first relation of Eqs. (2.8) implies that ϕ_1 does not depend on the coordinate along the Killing direction. On the other hand, we need to integrate the second equation so as to obtain the explicit form of ϕ_2. As a result of the integration, ϕ_2 generally has a form depending on the ignorable coordinate.

Note that the vector potential derived from Eq. (1.2) does not satisfy $\pounds_\zeta A_\mu=0$ even if the Euler potentials are chosen according to Eq. (2.8). In fact, from Eqs. (1.2) and (2.8), we have

$$\pounds_\zeta A_\mu=\pounds_\zeta\left[\frac{1}{2}(\phi_1\partial_\mu\phi_2-\phi_2\partial_\mu\phi_1)\right]=-\frac{1}{2}\partial_\mu\phi_1. \tag{2.23}$$

A gauge transformation $A_\mu\to\tilde{A}_\mu=A_\mu+\partial_\mu\lambda$ yields

$$\pounds_\zeta\tilde{A}_\mu=\pounds_\zeta A_\mu+\partial_\mu(\pounds_\zeta\lambda). \tag{2.24}$$

Thus if λ satisfies

$$\pounds_\zeta\lambda=\frac{1}{2}\phi_1, \tag{2.25}$$

the right-hand side of Eq. (2.24) vanishes. In this equation, ϕ_1 does not depend on the ignorable coordinate by virtue of $\pounds_\zeta\phi_1=0$. Therefore we can integrate it easily and get the explicit expression of λ.

B. Time-dependent axisymmetric configuration

In order to illustrate the method described above, let us consider the time-dependent axisymmetric configuration. For simplicity, we work in the flat space and use the spherical coordinate:

$$ds^2=-dt^2+dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{2.26}$$

The special case given by Eq. (2.9) is almost trivial. Thus we do not discuss this case further.

We assume that the force-free electromagnetic field is invariant under the action generated by the axial Killing vector $\mathbf{m}=\partial_\varphi$, that is,

$$\pounds_{\mathbf{m}}F_{\mu\nu}=0. \tag{2.27}$$

By Eqs. (2.8), the Euler potentials must satisfy

$$\pounds_{\mathbf{m}}\phi_1=\partial_\varphi\phi_1=0, \quad \pounds_{\mathbf{m}}\phi_2=\partial_\varphi\phi_2=1. \tag{2.28}$$

Integrating these equations, we have

$$\phi_1=\psi_1(t,r,\theta), \quad \phi_2=\varphi+\psi_2(t,r,\theta). \tag{2.29}$$

These are the general forms of the Euler potentials for the axisymmetric degenerate electromagnetic fields. Note that one of the Euler potentials explicitly depends on the axial coordinate φ even if the axial symmetry is assumed. The components of the electromagnetic field are

$$F_{tr}=\partial_t\psi_1\partial_r\psi_2-\partial_t\psi_2\partial_r\psi_1, \quad F_{t\theta}=\partial_t\psi_1\partial_\theta\psi_2-\partial_t\psi_2\partial_\theta\psi_1,$$

$$F_{t\varphi}=\partial_t\psi_1, \quad F_{r\varphi}=\partial_r\psi_1, \quad F_{\theta\varphi}=\partial_\theta\psi_1,$$

$$F_{r\theta}=\partial_r\psi_1\partial_\theta\psi_2-\partial_r\psi_2\partial_\theta\psi_1. \tag{2.30}$$

We have discussed the arbitrariness in the Euler potentials in paper I. The Euler potentials that yield a given axisymmetric electromagnetic field are not unique even after the Euler potentials are restricted to the form of Eq. (2.29). Evidently, ψ_1 is determined uniquely within the difference of a constant. On the other hand, by an arbitrary function of ψ_1, we can transform ψ_2 as

$$\psi_2\to\tilde{\psi}_2=\psi_2+f(\psi_1), \tag{2.31}$$

without changing the electromagnetic field. Obviously, this is an example of the transformation defined by Eqs. (2.5) and (2.6). The components of the vector potential (1.2) are

$$A_t=\frac{1}{2}(\psi_1\partial_t\psi_2-\psi_2\partial_t\psi_1)-\frac{1}{2}\varphi\partial_t\psi_1,$$

$$A_r=\frac{1}{2}(\psi_1\partial_r\psi_2-\psi_2\partial_r\psi_1)-\frac{1}{2}\varphi\partial_r\psi_1,$$

$$A_\theta=\frac{1}{2}(\psi_1\partial_\theta\psi_2-\psi_2\partial_\theta\psi_1)-\frac{1}{2}\varphi\partial_\theta\psi_1, \quad A_\varphi=\frac{1}{2}\psi_1. \tag{2.32}$$

This expression depends on the axial coordinate φ explicitly. Thus $\pounds_{\mathbf{m}}A_\mu\neq 0$ follows as already mentioned. The form of the vector potential satisfying $\pounds_{\mathbf{m}}A_\mu=0$ is derived by a gauge transformation

$$A_\mu\to\tilde{A}_\mu=A_\mu+\frac{1}{2}\partial_\mu(\varphi\psi_1), \tag{2.33}$$

according to Eq. (2.25). In fact, this transformation leads to

$$\tilde{A}_t=\frac{1}{2}(\psi_1\partial_t\psi_2-\psi_2\partial_t\psi_1), \quad \tilde{A}_r=\frac{1}{2}(\psi_1\partial_r\psi_2-\psi_2\partial_r\psi_1),$$

$$\tilde{A}_\theta=\frac{1}{2}(\psi_1\partial_\theta\psi_2-\psi_2\partial_\theta\psi_1), \quad \tilde{A}_\varphi=\psi_1. \tag{2.34}$$

Let $f(\psi_1,\psi_2)$ be an arbitrary function of ψ_1 and ψ_2. Then vector potentials derived from $\tilde{A}_\mu$ by gauge transformations such that $\tilde{A}_\mu\to\tilde{A}_\mu+\partial_\mu f(\psi_1,\psi_2)$ satisfy $\pounds_{\mathbf{m}}A_\mu=0$. Since $\partial_\varphi\psi_1=\partial_\varphi\psi_2=0$ holds, all the vector potentials satisfying $\pounds_{\mathbf{m}}A_\mu=0$ have the same φ component. Therefore $\psi_1=A_\mu\mathbf{m}^\mu$ is an invariant quantity among the vector potentials that yield the axisymmetric degenerate electromagnetic field and satisfy $\pounds_{\mathbf{m}}A_\mu=0$.

C. Dynamical description of time-dependent axisymmetric field configuration

1. Force-free equation

In the above discussions, we have seen that the condition for the symmetry restricts the form of the Euler potentials as Eq. (2.29). This equation gives the transformation of the variables from two Euler potentials (ϕ_1,ϕ_2) to another set of two scalars (ψ_1,ψ_2). Although one of the Euler potentials explicitly depends on the ignorable coordinate φ, new variables ψ_1 and ψ_2 do not depend on φ. Since there is no further condition for symmetry, generally we must treat two equations for two functions ψ_1 and ψ_2 in the field configuration invariant under the action of one symmetry group.

The electromagnetic field was already given by Eq. (2.30). We can decompose the magnetic field into the poloidal components and the toroidal component as

$$\vec{B}_{\rm pol}=\vec{\nabla}\psi_1\times\vec{e}_\varphi, \quad \vec{B}_{\rm tor}=\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2. \tag{2.35}$$

Similarly, the electric field is decomposed as

$$\vec{E}_{\rm pol}=-\dot{\psi}_1\vec{\nabla}\psi_2+\dot{\psi}_2\vec{\nabla}\psi_1, \quad \vec{E}_{\rm tor}=-\dot{\psi}_1\vec{e}_\varphi. \tag{2.36}$$

The decomposition of the electromagnetic field into the poloidal and the toroidal components arises naturally as in the traditional method. Substituting the Euler potentials (2.29) into the 3+1 form of the equation of motion [Eq. (4.2) of paper I], we have

$$\begin{pmatrix} \vec{\nabla}\psi_2\cdot\vec{\nabla}\psi_2+1/(r^2\sin^2\theta) & -\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2 \\ -\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2 & \vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1 \end{pmatrix}\begin{pmatrix}\ddot{\psi}_1\\ \ddot{\psi}_2\end{pmatrix}+\begin{pmatrix} \vec{\nabla}\dot{\psi}_2\cdot\vec{\nabla}\psi_2 & \vec{\nabla}\psi_1\cdot\vec{\nabla}\dot{\psi}_2-2\vec{\nabla}\dot{\phi}_1\cdot\vec{\nabla}\psi_2 \\ \vec{\nabla}\dot{\psi}_1\cdot\vec{\nabla}\psi_2-2\vec{\nabla}\psi_1\cdot\vec{\nabla}\dot{\psi}_2 & \vec{\nabla}\dot{\psi}_1\cdot\vec{\nabla}\psi_1 \end{pmatrix}\begin{pmatrix}\dot{\psi}_1\\ \dot{\psi}_2\end{pmatrix}$$

$$\pm(\dot{\psi}_2\vec{\nabla}^2\psi_1-\dot{\psi}_1\vec{\nabla}^2\psi_2)\begin{pmatrix}\dot{\psi}_2\\ \dot{\psi}_1\end{pmatrix}\pm\vec{\nabla}\times(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)\cdot\begin{pmatrix}\vec{\nabla}\psi_2\\ \vec{\nabla}\psi_1\end{pmatrix}-\begin{pmatrix}\vec{\nabla}\cdot[(1/r^2\sin^2\theta)\vec{\nabla}\psi_1]\\ 0\end{pmatrix}=0, \quad (2.37)$$

where $\vec{\nabla}$ is acting on the poloidal coordinates only. This equation describes the evolution of ψ_1 and ψ_2. If the condition

$$\begin{vmatrix} \vec{\nabla}\psi_2\cdot\vec{\nabla}\psi_2+1/(r^2\sin^2\theta) & -\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2 \\ -\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2 & \vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1 \end{vmatrix} =(\vec{\nabla}\phi_1\times\vec{\nabla}\phi_2)^2=|\vec{B}|^2\neq 0 \quad (2.38)$$

is satisfied, we can solve Eq. (2.37) for $\ddot{\psi}_1$ and $\ddot{\psi}_2$ as Eq. (4.4) of paper I. By virtue of the restricted forms of the Euler potentials, $\vec{\nabla}\phi_1\|\vec{\nabla}\phi_2$ and $\vec{\nabla}\phi_2=\vec{0}$ cannot occur. Consequently, only $\vec{\nabla}\psi_1=\vec{0}$ causes $|\vec{B}|=0$. Thus, as far as $\vec{\nabla}\psi_1\neq\vec{0}$ is satisfied in the whole force-free region, Eq. (2.37) determines the second time derivatives of ψ_1 and ψ_2 on an arbitrary three-space from $\psi_i(i=1,2)$ and their first time derivatives prescribed on this three-space.

Substituting Eq. (2.29) into the Lagrangian scalar (1.4), we have

$$L=\frac{1}{8\pi}\left[(\dot{\psi}_1\vec{\nabla}\psi_2-\dot{\psi}_2\vec{\nabla}\psi_1)^2+\frac{1}{r^2\sin^2\theta}\dot{\psi}_1^2-(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2 -\frac{1}{r^2\sin^2\theta}(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)\right]. \quad (2.39)$$

It is straightforward to see that the Euler-Lagrange equation

$$\frac{\partial L}{\partial\psi_i}-\partial_t\frac{\partial L}{\partial\dot{\psi}_i}-\vec{\nabla}\cdot\frac{\partial L}{\partial(\vec{\nabla}\psi_i)}=0, \quad i=1,2 \quad (2.40)$$

indeed yields Eq. (2.37).

2. Canonical equation of motion

The transformation from the Euler potentials (ϕ_1,ϕ_2) to a set of two scalars (ψ_1,ψ_2) is a canonical transformation. We gave the canonical formalism in paper I. Now, we consider the canonical equation of motion for $\psi_i(i=1,2)$. Since $\dot{\phi}_i=\dot{\psi}_i$, the canonical momentum conjugate to ψ_i is identical with π_i given by Eq. (5.21) of paper I. Thus we have

$$\pi_1=\frac{\partial L}{\partial\dot{\psi}_1} =\frac{1}{4\pi}\{(\vec{\nabla}\psi_2\cdot\vec{\nabla}\psi_2+1/r^2\sin^2\theta)\dot{\psi}_1-(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2)\dot{\psi}_2\},$$

$$\pi_2=\frac{\partial L}{\partial\dot{\psi}_2}=\frac{1}{4\pi}\{-(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2)\dot{\psi}_1+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)\dot{\psi}_2\}. \quad (2.41)$$

If $|\vec{B}|^2\neq 0$, we can solve the above equations for $\dot{\psi}_i$ as

$$\begin{pmatrix}\dot{\psi}_1\\ \dot{\psi}_2\end{pmatrix}=\frac{4\pi}{B^2}\begin{pmatrix}\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1 & \vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2\\ \vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_2 & \vec{\nabla}\psi_2\cdot\vec{\nabla}\psi_2+1/(r^2\sin^2\theta)\end{pmatrix}\begin{pmatrix}\pi_1\\ \pi_1\end{pmatrix}. \quad (2.42)$$

The Hamiltonian scalar becomes

$$H=\frac{1}{8\pi}\left\{\frac{(4\pi)^2[(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)^2+\pi_2^2/r^2\sin^2\theta]}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]} +(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta\right\}. \quad (2.43)$$

Then the canonical equations of motion are given by

$$\dot{\psi}_1=\frac{\delta H}{\delta\pi_1}=\frac{4\pi(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]}\cdot\vec{\nabla}\phi_1,$$

$$\dot{\psi}_2=\frac{\delta H}{\delta\pi_2}=\frac{4\pi(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]}\cdot\vec{\nabla}\phi_2, \quad (2.44)$$

and

$$\dot{\pi}_1=-\frac{\delta H}{\delta\psi_1} =\vec{\nabla}\cdot\left\{\frac{4\pi(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]}\pi_1 +\frac{1}{4\pi}\left[1-\frac{(4\pi)^2\{(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)^2+\pi_2^2/(r^2\sin^2\theta)\}}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]^2}\right] \times[\vec{\nabla}\psi_2\times(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)+(1/r^2\sin^2\theta)\vec{\nabla}\psi_1]\right\},$$

$$\dot{\pi}_2 = -\frac{\delta H}{\delta \psi_2}$$

$$= \vec{\nabla}\cdot\left\{\frac{4\pi(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]}\pi_2 - \frac{1}{4\pi}\left[1-\frac{(4\pi)^2\{(\pi_1\vec{\nabla}\psi_1+\pi_2\vec{\nabla}\psi_2)^2+\pi_2^2/(r^2\sin^2\theta)\}}{[(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)^2+(\vec{\nabla}\psi_1\cdot\vec{\nabla}\psi_1)/r^2\sin^2\theta]^2}\right] \times\vec{\nabla}\psi_1\times(\vec{\nabla}\psi_1\times\vec{\nabla}\psi_2)\right\}. \qquad (2.45)$$

These equations are longer than the corresponding canonical equations of motion in paper I apparently. However, actually they are simplified because $\vec{\nabla}$ becomes an operator acting on the poloidal coordinate only.

Equations (2.44) and (2.45) decide the causal development of axisymmetric force-free electromagnetic fields. Unfortunately, these are very complex equations. It is difficult to discuss solutions and physical contents of these equations at this stage. Thus we leave these matters untouched. However, our discussions are enough for showing the method for applying the general theory of the force-free electromagnetic field to the configuration with one direction of symmetry. Further, it is to be noted that in the foregoing works on the evolution of the axisymmetric force-free magnetosphere [2], any closed set of the basic equations for the evolving force-free electromagnetic field was not given. Equations (2.44) and (2.45) complement this major defect.

III. CONFIGURATION WITH TWO INVARIANT DIRECTIONS

A. Classification into two classes

Now, we are going to study the force-free electromagnetic field invariant under the action generated by two Killing vectors $\zeta^\mu_{(1)}$ and $\zeta^\mu_{(2)}$. We demand that the electromagnetic field satisfies

$$\pounds_{\zeta_{(1)}}F_{\mu\nu} = \pounds_{\zeta_{(2)}}F_{\mu\nu} = 0. \qquad (3.1)$$

First, we consider the general condition on the Euler potentials that yields the degenerate electromagnetic fields satisfying Eq. (3.1). The same reasoning as the configuration with one symmetry direction also holds. Thus, corresponding to Eq. (2.12), two relations,

$$F_{\mu\lambda}\zeta^\lambda_{(1)} = \partial_\mu f_{(1)}(\phi_1,\phi_2), \quad F_{\mu\lambda}\zeta^\lambda_{(2)} = \partial_\mu f_{(2)}(\phi_1,\phi_2), \qquad (3.2)$$

follow from Eq. (3.1). Here $f_{(1)}(\phi_1,\phi_2)$ and $f_{(2)}(\phi_1,\phi_2)$ are two arbitrary functions of ϕ_1 and ϕ_2. As Eq. (2.14) results from Eq. (2.12), from Eqs. (3.2), we have

$$\zeta^\mu_{(1)}\partial_\mu\phi_1 = -\frac{\partial f_{(1)}}{\partial\phi_2}, \quad \zeta^\mu_{(1)}\partial_\mu\phi_2 = \frac{\partial f_{(1)}}{\partial\phi_1}, \qquad (3.3)$$

and

$$\zeta^\mu_{(2)}\partial_\mu\phi_1 = -\frac{\partial f_{(2)}}{\partial\phi_2}, \quad \zeta^\mu_{(2)}\partial_\mu\phi_2 = \frac{\partial f_{(2)}}{\partial\phi_1}. \qquad (3.4)$$

These relations are also covariantly transformed as Eq. (2.14) under the transformation defined by Eqs. (2.5) and (2.6). Further, from Eqs. (3.3) and (3.4), we find

$$F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)} = (\partial_\mu\phi_1\partial_\lambda\phi_2 - \partial_\mu\phi_2\partial_\lambda\phi_1)\zeta^\mu_{(1)}\zeta^\lambda_{(2)} = \frac{\partial(f_{(1)},f_{(2)})}{\partial(\phi_1,\phi_2)}. \qquad (3.5)$$

Thus we should treat the two cases (1) $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}=0$ and (2) $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}\neq 0$ separately.

These two cases are also distinguished by the existence of the generator of the flux surface written by a linear combination of two Killing vectors. Namely, when $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}=0$ holds, there exists a generator of the flux surface written by a linear combination of two Killing vectors. On the other hand, when $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}\neq 0$, there does not exist a generator of the flux surface written by a linear combination of two Killing vectors. Inversely, if a degenerate electromagnetic field invariant under the action of two Killing vectors $\zeta^\mu_{(1)}$ and $\zeta^\lambda_{(2)}$ has a generator of the flux surface written by a linear combination of these two Killing vectors, $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}=0$ follows. A proof is as follows. Let ξ^μ be a generator of the flux surface written by a linear combination of two Killing vectors. That is,

$$\xi^\mu = a^{(1)}\zeta^\mu_{(1)} + a^{(2)}\zeta^\mu_{(2)}, \qquad (3.6)$$

where $a^{(1)}$ and $a^{(2)}$ may be functions of position. Here, $F_{\mu\nu}\xi^\mu=0$ holds because ξ^μ is a generator of the flux surface by the assumption. By Eq. (1.1), we have

$$F_{\mu\nu}\xi^\nu = (a^{(1)}\zeta^\nu_{(1)}\partial_\nu\phi_2 + a^{(2)}\zeta^\nu_{(2)}\partial_\nu\phi_2)\partial_\mu\phi_1 - (a^{(1)}\zeta^\nu_{(1)}\partial_\nu\phi_1 + a^{(2)}\zeta^\nu_{(2)}\partial_\nu\phi_1)\partial_\mu\phi_2. \qquad (3.7)$$

As long as $F_{\mu\nu}$ does not vanish, $\partial_\mu\phi_1$ and $\partial_\mu\phi_2$ are two linearly independent vectors. Thus nontrivial ξ^ν satisfying $F_{\mu\nu}\xi^\nu=0$ exists if and only if the equation

$$\begin{pmatrix} \zeta^\nu_{(1)}\partial_\nu\phi_2 & \zeta^\nu_{(2)}\partial_\nu\phi_2 \\ -\zeta^\nu_{(1)}\partial_\nu\phi_1 & -\zeta^\nu_{(2)}\partial_\nu\phi_1 \end{pmatrix}\begin{pmatrix} a^{(1)} \\ a^{(2)} \end{pmatrix} = 0 \qquad (3.8)$$

has a nontrivial solution of $(a^{(1)},a^{(2)})$. Therefore a generator of the flux surface written by a linear combination of two Killing vectors exists if and only if

$$\begin{vmatrix} \zeta^\nu_{(1)}\partial_\nu\phi_2 & \zeta^\nu_{(2)}\partial_\nu\phi_2 \\ -\zeta^\nu_{(1)}\partial_\nu\phi_1 & -\zeta^\nu_{(2)}\partial_\nu\phi_1 \end{vmatrix} = (\partial_\mu\phi_1\partial_\nu\phi_2 - \partial_\mu\phi_2\partial_\nu\phi_1)\zeta^\mu_{(1)}\zeta^\nu_{(2)} = F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)} = 0 \qquad (3.9)$$

is satisfied. Inversely, it is evident that a generator of the flux surface ξ^μ written as Eq. (3.6) exists when Eq. (3.9) holds. Therefore if $F_{\mu\lambda}\zeta^\mu_{(1)}\zeta^\lambda_{(2)}=0$, there is a generator of the flux surface written by a linear combination of two Killing vectors. We refer to this case as case I henceforth. On the other hand, if $F_{\mu\lambda}\zeta^{\mu}_{(1)}\zeta^{\lambda}_{(2)}\neq 0$, there does not exist a generator of the flux surface written by a linear combination of two Killing vectors. We refer to this case as case II.

B. Euler Potentials in case I

Let us assume $F_{\mu\lambda}\zeta^{\mu}_{(1)}\zeta^{\lambda}_{(2)}=0$. By Eq. (3.5), this implies

$$\frac{\partial f_{(1)}}{\partial\phi_1}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial f_{(2)}}{\partial\phi_1}\frac{\partial f_{(1)}}{\partial\phi_2}=0. \tag{3.10}$$

By means of the same procedure as described in the preceding section, we can construct a new set of the Euler potentials $(\widetilde{\phi}_1,\widetilde{\phi}_2)$ such that

$$\widetilde{\phi}_1=f_{(1)}(\phi_1,\phi_2),\quad \frac{\partial(\widetilde{\phi}_1,\widetilde{\phi}_2)}{\partial(\phi_1,\phi_2)}=1. \tag{3.11}$$

As Eq. (2.22), then we have

$$\frac{\partial f_{(1)}}{\partial\widetilde{\phi}_1}=1,\quad \frac{\partial f_{(1)}}{\partial\widetilde{\phi}_2}=0. \tag{3.12}$$

Further, from the second of Eqs. (3.11), we have $\partial(f_{(1)},f_{(2)})/\partial(\phi_1,\phi_2)=\partial(f_{(1)},f_{(2)})/\partial(\widetilde{\phi}_1,\widetilde{\phi}_2)$. Thus Eqs. (3.10) and (3.12) imply

$$\frac{\partial f_{(2)}}{\partial\widetilde{\phi}_2}=0. \tag{3.13}$$

Therefore we have

$$f_{(2)}=f_{(2)}(\widetilde{\phi}_1). \tag{3.14}$$

Since $f_{(2)}$ is a function of $\widetilde{\phi}_1$, we can also write

$$\frac{\partial f_{(2)}}{\partial\widetilde{\phi}_1}=\kappa(\widetilde{\phi}_1). \tag{3.15}$$

After substituting Eqs. (3.12), (3.13), and (3.15) into Eqs. (3.3) and (3.4), it turns out that Euler potentials $\widetilde{\phi}_1$ and $\widetilde{\phi}_1$ satisfy

$$\pounds_{\zeta_{(1)}}\widetilde{\phi}_1=\zeta^{\mu}_{(1)}\partial_\mu\widetilde{\phi}_1=0,\quad \pounds_{\zeta_{(2)}}\widetilde{\phi}_1=\zeta^{\mu}_{(2)}\partial_\mu\widetilde{\phi}_1=0, \tag{3.16}$$

and

$$\pounds_{\zeta_{(1)}}\widetilde{\phi}_2=\zeta^{\mu}_{(1)}\partial_\mu\widetilde{\phi}_2=1,\quad \pounds_{\zeta_{(2)}}\widetilde{\phi}_2=\zeta^{\mu}_{(2)}\partial_\mu\widetilde{\phi}_1=\kappa(\widetilde{\phi}_1), \tag{3.17}$$

respectively. We have not imposed any condition on the functional form of $f_{(1)}$ and $f_{(2)}$ so far. Thus the above discussion is general. Accordingly, it is always possible to find Euler potentials obeying Eqs. (3.16) and (3.17). Therefore the Euler potentials for the degenerate electromagnetic field having two directions of symmetry and satisfying $F_{\mu\nu}\zeta^{\mu}_{(1)}\zeta^{\nu}_{(2)}=0$ are determined by the conditions

$$\pounds_{\zeta_{(1)}}\phi_1=\zeta^{\mu}_{(1)}\partial_\mu\phi_1=0,\quad \pounds_{\zeta_{(2)}}\phi_1=\zeta^{\mu}_{(2)}\partial_\mu\phi_1=0,$$
$$\pounds_{\zeta_{(1)}}\phi_2=\zeta^{\mu}_{(1)}\partial_\mu\phi_2=1,\quad \pounds_{\zeta_{(2)}}\phi_2=\zeta^{\mu}_{(2)}\partial_\mu\phi_1=\kappa(\phi_1), \tag{3.18}$$

without any loss of generality. These equations are integrated easily if the Killing vectors are explicitly given. From two of them, i.e., $\pounds_{\zeta_{(1)}}\phi_2=1$ and $\pounds_{\zeta_{(2)}}\phi_2=\kappa(\phi_1)$, we have an expression of ϕ_2 depending on the ignorable coordinates.

Sometimes we can see that it is sufficient to treat case I only by some reasons. For example, in the stationary and axisymmetric configuration, we can exclude case II demanding regularity of the electromagnetic field at the pole. In such cases, we can derive conditions (3.18) more easily assuming $\pounds_{\zeta_{(1)}}\phi_1=0$ only. As discussed in the preceding section, $F_{\mu\nu}\zeta^{\nu}_{(1)}=0$ is a too strong condition. Together with $F_{\mu\lambda}\zeta^{\mu}_{(1)}\zeta^{\lambda}_{(2)}=0$, this leads to the condition $\pounds_{\zeta_{(2)}}\phi_1=0$. Then Eqs. (3.1) become

$$\pounds_{\zeta_{(1)}}F_{\mu\nu}=\partial_\mu\phi_1\partial_\nu(\pounds_{\zeta_{(1)}}\phi_2)-\partial_\mu(\pounds_{\zeta_{(1)}}\phi_2)\partial_\mu\phi_1=0,$$
$$\pounds_{\zeta_{(2)}}F_{\mu\nu}=\partial_\mu\phi_1\partial_\nu(\pounds_{\zeta_{(2)}}\phi_2)-\partial_\mu(\pounds_{\zeta_{(2)}}\phi_2)\partial_\mu\phi_1=0. \tag{3.19}$$

For the same reason that Eq. (2.4) arises from condition (2.2), we find

$$\pounds_{\zeta_{(1)}}\phi_2=f_1(\phi_1),\quad \pounds_{\zeta_{(2)}}\phi_2=f_2(\phi_1), \tag{3.20}$$

where $f_1(\phi_1)$ and $f_2(\phi_1)$ are two arbitrary functions of ϕ_1. Thus the same transformation as Eq. (2.7) makes the Euler potentials satisfy either $f_1(\phi_1)=1$ or $f_2(\phi_1)=1$. However, generally we cannot set two Euler potentials to 1 simultaneously. Therefore one integral on the three-surface of constant ϕ_1 appears. Thus conditions (3.18) determine the Euler potentials.

Substituting the Euler potentials satisfying Eqs. (3.18) into Eq. (1.2), we have an expression of the vector potential. However, this satisfies neither $\pounds_{\zeta_{(1)}}A_\mu=0$ nor $\pounds_{\zeta_{(2)}}A_\mu=0$. In fact, we have

$$\pounds_{\zeta_{(1)}}A_\mu=-\frac{1}{2}\partial_\mu\phi_1,$$
$$\pounds_{\zeta_{(2)}}A_\mu=\frac{1}{2}\left(\phi_1\frac{d\kappa(\phi_1)}{d\phi_1}-\kappa(\phi_1)\right)\partial_\mu\phi_1 = \partial_\mu\left[\frac{1}{2}\int\left(\phi_1\frac{d\kappa(\phi_1)}{d\phi_1}-\kappa(\phi_1)\right)d\phi_1\right]. \tag{3.21}$$

However, a gauge transformation $A_\mu\rightarrow\widetilde{A}_\mu=A_\mu+\partial_\mu\lambda$ makes the vector potential satisfy $\pounds_{\zeta_{(1)}}\widetilde{A}_\mu=\pounds_{\zeta_{(2)}}\widetilde{A}_\mu=0$, if λ satisfies

$$\pounds_{\zeta_{(1)}}\lambda=\frac{1}{2}\phi_1,$$

$$\pounds_{\zeta_{(2)}}\lambda=-\frac{1}{2}\int\left(\phi_1\frac{d\kappa(\phi_1)}{d\phi_1}-\kappa(\phi_1)\right)d\phi_1$$
$$=-\frac{1}{2}\kappa(\phi_1)\phi_1+\int\kappa(\phi_1)d\phi_1. \quad (3.22)$$

Since ϕ_1 does not depend on two ignorable coordinates, Eqs. (3.22) are integrable.

C. Euler potentials in case II

Let us consider case II. For simplicity, we restrict our consideration to the space-time in which two Killing vectors $\zeta^{\mu}_{(1)}$ and $\zeta^{\lambda}_{(2)}$ commute with each other. Namely, we assume

$$\pounds_{\zeta_{(1)}}\zeta_{(2)}=-\pounds_{\zeta_{(2)}}\zeta_{(1)}=[\zeta_{(1)},\zeta_{(2)}]=0, \quad (3.23)$$

where $[\zeta_{(1)},\zeta_{(2)}]$ is the commutator of $\zeta_{(1)}$ and $\zeta_{(2)}$. From Eq. (3.23), we have two identities:

$$\zeta^{\nu}_{(1)}\partial_\nu(\zeta^{\mu}_{(2)}\partial_\mu\phi_1)-\zeta^{\nu}_{(2)}\partial_\nu(\zeta^{\mu}_{(1)}\partial_\mu\phi_1)=0,$$
$$\zeta^{\nu}_{(1)}\partial_\nu(\zeta^{\mu}_{(2)}\partial_\mu\phi_2)-\zeta^{\nu}_{(2)}\partial_\nu(\zeta^{\mu}_{(1)}\partial_\mu\phi_2)=0. \quad (3.24)$$

Substituting Eqs. (3.3) and (3.4) into the above equations, we get

$$\frac{\partial^2 f_{(2)}}{\partial\phi_1\partial\phi_2}\zeta^{\mu}_{(1)}\partial_\mu\phi_1+\frac{\partial^2 f_{(2)}}{\partial\phi_2^2}\zeta^{\mu}_{(1)}\partial_\mu\phi_2-\frac{\partial^2 f_{(1)}}{\partial\phi_1\partial\phi_2}\zeta^{\mu}_{(2)}\partial_\mu\phi_1$$
$$-\frac{\partial^2 f_{(1)}}{\partial\phi_2^2}\zeta^{\mu}_{(2)}\partial_\mu\phi_2=0 \quad (3.25)$$

and

$$\frac{\partial^2 f_{(2)}}{\partial\phi_1^2}\zeta^{\mu}_{(1)}\partial_\mu\phi_1+\frac{\partial^2 f_{(2)}}{\partial\phi_1\partial\phi_2}\zeta^{\mu}_{(1)}\partial_\mu\phi_2-\frac{\partial^2 f_{(1)}}{\partial\phi_1^2}\zeta^{\mu}_{(2)}\partial_\mu\phi_1$$
$$-\frac{\partial^2 f_{(1)}}{\partial\phi_1\partial\phi_2}\zeta^{\mu}_{(2)}\partial_\mu\phi_2=0. \quad (3.26)$$

Further, substituting Eqs. (3.3) and (3.4) into the above equations again, we have

$$\frac{\partial^2 f_{(1)}}{\partial\phi_1\partial\phi_2}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial^2 f_{(1)}}{\partial\phi_2^2}\frac{\partial f_{(2)}}{\partial\phi_1}-\frac{\partial^2 f_{(2)}}{\partial\phi_1\partial\phi_2}\frac{\partial f_{(1)}}{\partial\phi_2}+\frac{\partial^2 f_{(2)}}{\partial\phi_2^2}\frac{\partial f_{(1)}}{\partial\phi_1}$$
$$=\frac{\partial}{\partial\phi_2}\left[\frac{\partial f_{(1)}}{\partial\phi_1}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial f_{(1)}}{\partial\phi_2}\frac{\partial f_{(2)}}{\partial\phi_1}\right]=0 \quad (3.27)$$

and

$$\frac{\partial^2 f_{(1)}}{\partial\phi_1^2}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial^2 f_{(1)}}{\partial\phi_1\partial\phi_2}\frac{\partial f_{(2)}}{\partial\phi_1}-\frac{\partial^2 f_{(2)}}{\partial\phi_1^2}\frac{\partial f_{(1)}}{\partial\phi_2}+\frac{\partial^2 f_{(2)}}{\partial\phi_1\partial\phi_2}\frac{\partial f_{(1)}}{\partial\phi_1}$$
$$=\frac{\partial}{\partial\phi_1}\left[\frac{\partial f_{(1)}}{\partial\phi_1}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial f_{(1)}}{\partial\phi_2}\frac{\partial f_{(2)}}{\partial\phi_1}\right]=0. \quad (3.28)$$

Here the Jacobian $\partial(f_{(1)},f_{(2)})/\partial(\phi_1,\phi_2)$ is generally a function of ϕ_1 and ϕ_2. Thus Eqs. (3.27) and (3.28) imply

$$\frac{\partial f_{(1)}}{\partial\phi_1}\frac{\partial f_{(2)}}{\partial\phi_2}-\frac{\partial f_{(1)}}{\partial\phi_2}\frac{\partial f_{(2)}}{\partial\phi_1}=\frac{\partial(f_{(1)},f_{(2)})}{(\phi_1,\phi_2)}=\kappa, \quad (3.29)$$

where κ is a constant. Further, we can introduce a new set of Euler potentials as

$$\tilde{\phi}_1=f_{(1)}(\phi_1,\phi_2), \quad \tilde{\phi}_2=\frac{1}{\kappa}f_{(2)}(\phi_1,\phi_2). \quad (3.30)$$

Evidently, Eq. (2.6) holds. Thus $\tilde{\phi}_1$ and $\tilde{\phi}_2$ satisfy

$$\zeta^{\mu}_{(1)}\partial_\mu\tilde{\phi}_1=-\frac{\partial f_{(1)}}{\partial\tilde{\phi}_2}=0, \quad \zeta^{\mu}_{(1)}\partial_\mu\tilde{\phi}_2=\frac{\partial f_{(1)}}{\partial\tilde{\phi}_1}=1, \quad (3.31)$$

and

$$\zeta^{\mu}_{(2)}\partial_\mu\tilde{\phi}_1=-\frac{\partial f_{(2)}}{\partial\tilde{\phi}_2}=\kappa, \quad \zeta^{\mu}_{(2)}\partial_\mu\tilde{\phi}_2=\frac{\partial f_{(2)}}{\partial\tilde{\phi}_1}=0. \quad (3.32)$$

In the above discussion, we do not impose any restriction on the functional forms of $f_{(1)}$ and $f_{(2)}$. Therefore, in case II, the Euler potentials are generally determined by the conditions

$$\pounds_{\zeta_{(1)}}\phi_1=\zeta^{\mu}_{(1)}\partial_\mu\phi_1=0, \quad \pounds_{\zeta_{(2)}}\phi_1=\zeta^{\mu}_{(2)}\partial_\mu\phi_1=\kappa,$$
$$\pounds_{\zeta_{(1)}}\phi_2=\zeta^{\mu}_{(1)}\partial_\mu\phi_2=1, \quad \pounds_{\zeta_{(2)}}\phi_2=\zeta^{\mu}_{(2)}\partial_\mu\phi_1=0. \quad (3.33)$$

The vector potential introduced by Eq. (1.2) satisfies

$$\pounds_{\zeta_{(1)}}A_\mu=\frac{1}{2}\partial_\mu\phi_2, \quad \pounds_{\zeta_{(2)}}A_\mu=-\frac{1}{2}\kappa\partial_\mu\phi_1, \quad (3.34)$$

when the Euler potentials satisfy Eq. (3.33). So that a gauge transformation $A_\mu\to\tilde{A}_\mu=A_\mu+\partial_\mu\lambda$ makes the vector potential satisfy $\pounds_{\zeta_{(1)}}\tilde{A}_\mu=\pounds_{\zeta_{(2)}}\tilde{A}_\mu=0$, λ must satisfy

$$\pounds_{\zeta_{(1)}}\lambda=-\frac{1}{2}\phi_2, \quad \pounds_{\zeta_{(2)}}\lambda=\frac{1}{2}\kappa\phi_1. \quad (3.35)$$

However, this implies

$$\pounds_{\zeta_{(2)}}\pounds_{\zeta_{(1)}}\lambda=-\frac{1}{2}\kappa, \quad \pounds_{\zeta_{(1)}}\pounds_{\zeta_{(2)}}\lambda=\frac{1}{2}\kappa. \quad (3.36)$$

Thus Eqs. (3.35) are incompatible with each other. Namely, Eqs. (3.35) are not integrable as far as $\kappa\neq 0$. ($\kappa=0$ implies $F_{\mu\nu}\zeta^{\mu}_{(1)}\zeta^{\nu}_{(2)}=0$.) In contrast to case I, the vector potential satisfying both $\pounds_{\zeta_{(1)}}A_\mu=0$ and $\pounds_{\zeta_{(2)}}A_\mu=0$ does not exist in case II.

D. Stationary and axisymmetric configuration

The most important application of the force-free electromagnetic field with two invariant directions will be the stationary and axisymmetric configuration. It has already been studied extensively [1,3]. Thus this configuration is appropriate to clarify the relation between our formalism and the traditional method, although the results obtained are not new.

Therefore let us consider the stationary and axisymmetric force-free electromagnetic field.

By the assumption, the force-free electromagnetic field satisfies the conditions

$$\pounds_{\mathbf{k}}F_{\mu\nu}=0, \quad \pounds_{\mathbf{m}}F_{\mu\nu}=0, \tag{3.37}$$

where $\mathbf{k}$ is the stationary Killing vector ∂_t. As described in the foregoing subsection, two cases arise. In case I, the electromagnetic field satisfies $F_{\mu\nu}\mathbf{k}^\mu\mathbf{m}^\nu=F_{t\varphi}=0$. On the other hand, in case II, $F_{t\varphi}\neq 0$ holds. The former is the case studied in the foregoing works. We will treat this case in the following. In the latter case, however, the toroidal component of the electric field becomes singular at the pole as seen below. Thus it cannot be a solution appropriate to describe the whole region of the magnetosphere. However, since this paper aims at illustrating the method, we treat this case in the last subsection briefly. We work in the flat space-time with the spherical coordinate (2.26) again.

1. Euler potentials and electromagnetic field

Substituting the explicit forms of the Killing vectors into Eqs. (3.18), we can integrate these equations as

$$\phi_1=\Psi_1(r,\theta), \quad \phi_2=\varphi-\Omega_F(\Psi_1)t+\Psi_2(r,\theta). \tag{3.38}$$

In the above equations, we rewrite $\kappa(\Psi_1)$ to $-\Omega_F(\Psi_1)$ so as to agree with the conventional notation. Further, Ψ_1 is introduced to make the notation symmetric. These are the general forms of the Euler potentials. The components of the electromagnetic field then become

$$F_{tr}=\Omega_F\partial_r\Psi_1, \quad F_{t\theta}=\Omega_F\partial_\theta\Psi_1, \quad F_{t\varphi}=0, \quad F_{r\varphi}=\partial_r\Psi_1,$$

$$F_{\theta\varphi}=\partial_\theta\Psi_1, \quad F_{r\theta}=\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1. \tag{3.39}$$

Equations (3.39) show that our Ψ_1 is equivalent to the stream function of the poloidal magnetic field in the traditional method. Similarly, we can also see that $\Omega_F(\Psi_1)$ indeed corresponds to the angular velocity of the poloidal magnetic field lines [1,3].

Note that Ψ_2 appears only through $F_{r\theta}$. Further, from Eq. (3.39), we can see that Ψ_2 contains indeterminacy. It is determined within the arbitrariness arising from the transformation as

$$\Psi_2\to\Psi_2+f(\Psi_1), \tag{3.40}$$

where $f(\Psi_1)$ is an arbitrary function of Ψ_1. The vector potential defined by Eq. (1.2) becomes

$$A_t=-\frac{1}{2}\Omega_F\Psi_1,$$

$$A_r=\frac{1}{2}(\Psi_1\partial_r\Psi_2-\Psi_2\partial_r\Psi_1)+\frac{1}{2}\left[\left(\Omega_F-\frac{d\Omega_F}{d\Psi_1}\Psi_1\right)t-\varphi\right]\partial_r\Psi_1,$$

$$A_\theta=\frac{1}{2}(\Psi_1\partial_\theta\Psi_2-\Psi_2\partial_\theta\Psi_1)+\frac{1}{2}\left[\left(\Omega_F-\frac{d\Omega_F}{d\Psi_1}\Psi_1\right)t-\varphi\right]\partial_\theta\Psi_1,$$

$$A_\varphi=\frac{1}{2}\Psi_1. \tag{3.41}$$

We have an expression depending on the ignorable coordinates t and φ. This implies $\pounds_{\mathbf{k}}A_\mu\neq 0$ and $\pounds_{\mathbf{m}}A_\mu\neq 0$. According to Eq. (3.22), the expression that satisfies $\pounds_{\mathbf{k}}A_\mu=0$ and $\pounds_{\mathbf{m}}A_\mu=0$ is obtained by a gauge transformation

$$A_\mu\to\tilde{A}_\mu=A_\mu+\partial_\mu\lambda(r,\theta), \tag{3.42}$$

with

$$\lambda=\frac{1}{2}\Psi_1\varphi+\left[\frac{1}{2}\Omega_F(\Psi_1)\Psi_1-\int\Omega_F(\Psi_1)d\Psi_1\right]t. \tag{3.43}$$

Indeed, $\tilde{A}_\mu$ becomes

$$\tilde{A}_t=-\int\Omega_F(\Psi_1)d\Psi_1, \quad \tilde{A}_r=\frac{1}{2}(\Psi_1\partial_r\Psi_2-\Psi_2\partial_r\Psi_1),$$

$$\tilde{A}_\theta=\frac{1}{2}(\Psi_1\partial_\theta\Psi_2-\Psi_2\partial_\theta\Psi_1), \quad \tilde{A}_\varphi=\Psi_1. \tag{3.44}$$

In this expression, Ψ_1 becomes the φ component of the vector potential and hence coincides with the stream function of the poloidal magnetic field lines in the traditional method. Suppose a gauge transformation such that $\tilde{A}_\mu\to\tilde{A}_\mu+\partial_\mu f(\Psi_1,\Psi_2)$. The vector potentials derived from $\tilde{A}_\mu$ by this gauge transformation all satisfy $\pounds_{\mathbf{k}}A_\mu=0$ and $\pounds_{\mathbf{m}}A_\mu=0$. Under these gauge transformations $A_t=A_\mu\mathbf{k}^\mu$ and $A_\varphi=\Psi_1=A_\mu\mathbf{m}^\mu$ are invariant. Namely, A_t and A_φ are invariant quantities among the vector potentials satisfying $\pounds_{\mathbf{k}}A_\mu=0$ and $\pounds_{\mathbf{m}}A_\mu=0$.

2. Force-free equation

Let us consider the force-free condition. For illustrative purposes, first, we are going to summarize the traditional treatment briefly and then treat the same problem by the present formalism.

Since $\partial_t=\partial_\varphi=0$, J^r and J^θ are given by

$$4\pi\sqrt{-g}J^r=\partial_\theta(\sqrt{-g}F^{r\theta}), \quad 4\pi\sqrt{-g}J^\theta=-\partial_r(\sqrt{-g}F^{r\theta}), \tag{3.45}$$

respectively, where $\sqrt{-g}=r^2\sin\theta$. From $F_{t\varphi}=0$, t and φ components of the force-free equation, we have

$$\partial_r\Psi_1\partial_\theta(\sqrt{-g}F^{r\theta})-\partial_\theta\Psi_1\partial_r(\sqrt{-g}F^{r\theta})=0. \tag{3.46}$$

This implies that $\sqrt{-g}F^{r\theta}$ is a function of Ψ_1. That is,

$$\sqrt{-g}F^{r\theta}=r^2\sin\theta F^{r\theta}=B_T(\Psi_1). \tag{3.47}$$

Then the poloidal components of the current are written as

$$J^r=\frac{1}{4\pi\sqrt{-g}}\frac{dB_T}{d\Psi_1}\partial_\theta\Psi_1,\quad J^\theta=-\frac{1}{4\pi\sqrt{-g}}\frac{dB_T}{d\Psi_1}\partial_r\Psi_1. \tag{3.48}$$

Substituting Eqs. (3.48) into r and θ components of the force-free equation, we have

$$\left\{J^\varphi-\Omega_F(\Psi_1)J^t-\frac{B_T}{4\pi r^2\sin^2\theta}\frac{dB_T}{d\Psi_1}\right\}\partial_A\Psi_1=0, \tag{3.49}$$

where $A=r,\theta$. Thus we have

$$J^\varphi-\Omega_F(\Psi_1)J^t-\frac{B_T}{4\pi r^2\sin^2\theta}\frac{dB_T}{d\Psi_1}=0. \tag{3.50}$$

By straightforward calculations, J^t and J^φ become

$$J^t=-\frac{1}{4\pi\sqrt{-g}}\left[\partial_r(r^2\sin\theta\Omega_F\partial_r\Psi_1)+\frac{1}{r}\partial_\theta\left(r^2\sin\theta\Omega_F\frac{1}{r}\partial_\theta\Psi_1\right)\right] \tag{3.51}$$

and

$$J^\varphi=-\frac{1}{4\pi\sqrt{-g}}\left[\partial_r\left(\frac{1}{\sin\theta}\partial_r\Psi_1\right)+\frac{1}{r}\partial_\theta\left(\frac{1}{\sin\theta}\frac{1}{r}\partial_\theta\Psi_1\right)\right]. \tag{3.52}$$

Thus Eq. (3.50) becomes

$$\vec{\nabla}\cdot\left[\frac{1}{\varpi^2}(1-\varpi^2\Omega_F^2)\vec{\nabla}\Psi_1\right]+\Omega_F\frac{d\Omega_F}{d\Psi_1}\vec{\nabla}\Psi_1\cdot\vec{\nabla}\Psi_1+\frac{B_T}{\varpi^2}\frac{dB_T}{d\Psi_1}=0, \tag{3.53}$$

where ϖ is the cylindrical radius, i.e., $\varpi=r\sin\theta$. This is the well-known pulsar equation (transfield equation) [1–3] that decides the structure of the stationary and axisymmetric force-free magnetosphere.

Note that Ψ_2 does not appear at all in the above discussion. Evidently, this is because Ψ_2 appears only through $F_{r\theta}$ in the electromagnetic field as Eq. (3.39), and $F_{r\theta}$ itself is expressed by a function of Ψ_1 as Eq. (3.47). Indeed Eq. (3.47) is also written as

$$\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1=\frac{1}{\sin\theta}B_T(\Psi_1). \tag{3.54}$$

This is the equation for Ψ_2. It decides Ψ_2 within the arbitrariness of Eq. (3.40). However, in the traditional approach, the integral $B_T(\Psi_1)$ eliminates Ψ_2. Thus we can complete the whole analysis on the stationary and axisymmetric force-free electromagnetic field without mentioning Ψ_2 at all. This is the reason the traditional method can describe the stationary and axisymmetric configuration by one stream function. Equation (3.54) is required only when we want the information on the magnetic field lines or the flux surface.

Next let us treat the same problem by the present formalism. Substituting the Euler potentials (3.38) into the basic equations, we have two equations for Ψ_1 and Ψ_2. However, the same equations are derived from the action principle more easily. Substituting Eqs. (3.38) into the Lagrangian density $\mathcal{L}=\sqrt{-g}L$, we have

$$\mathcal{L}=-\frac{r^2\sin\theta}{8\pi}\left\{\frac{1}{\varpi^2}(1-\varpi^2\Omega_F^2)\left[(\partial_r\Psi_1)^2+\frac{1}{r^2}(\partial_\theta\Psi_1)^2\right]+\frac{1}{r^2}(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)^2\right\}. \tag{3.55}$$

The Euler-Lagrange equation for Ψ_1 yields

$$\begin{aligned}&\partial_r\left[\frac{1}{\sin\theta}(1-\varpi^2\Omega_F)\partial_r\Psi_1\right]+\frac{1}{r}\partial_\theta\left[\frac{1}{\sin\theta}(1-\varpi^2\Omega_F)\frac{1}{r}\partial_\theta\Psi_1\right]\\&+r^2\sin\theta\Omega_F\frac{d\Omega_F}{d\Psi_1}\left[(\partial_r\Psi_1)^2+\frac{1}{r^2}(\partial_\theta\Psi_1)^2\right]\\&+\partial_r[\sin\theta(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)]\partial_\theta\Psi_2\\&-\partial_\theta[\sin\theta(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)]\partial_r\Psi_2=0.\end{aligned} \tag{3.56}$$

Dividing Eq. (3.56) by $r^2\sin\theta$, we have an equation identical with Eq. (3.53) except for the last term. The Euler-Lagrange equation for Ψ_2 leads to

$$\begin{aligned}&\partial_\theta[\sin\theta(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)]\partial_r\Psi_1\\&-\partial_r[\sin\theta(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)]\partial_\theta\Psi_1=0.\end{aligned} \tag{3.57}$$

This equation is identical with Eq. (3.46). We can integrate this equation and have Eq. (3.54). Substituting integral $B_T(\Psi_1)$ into the last two terms of Eq. (3.56), it becomes

$$\frac{dB_T}{d\Psi_1}(\partial_r\Psi_1\partial_\theta\Psi_2-\partial_r\Psi_2\partial_\theta\Psi_1)=\frac{B_T}{\sin\theta}\frac{dB_T}{d\Psi_1}. \tag{3.58}$$

Thus we get the pulsar equation (3.53).

From the above discussion, we can understand general features of the force-free electromagnetic field with two invariant directions satisfying $F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)}=0$. In this type of field configurations, one of two basic equations is integrable. As a result, one of two Euler potentials is hidden behind the integral written by the other Euler potential. Further, one integral is already contained in one of the Euler potentials as Eq. (3.38). Consequently we have two integrals. Therefore this type of force-free electromagnetic field can be described by one Euler potential together with two integrals written by this Euler potential. The other Euler potential is required only in problems that are concerned with the flux surface or the magnetic field line.

3. Approach to stationary state

The Euler potential ϕ_2 depends on time even in the stationary and axisymmetric configuration as Eqs. (3.38). Thus ψ_2, one of the dynamical variables in the time-dependent axisymmetric configuration, has a specific time-dependent form in the stationary limit. Thus the stationary axisymmetric limit is not realized by setting the first and the second time derivatives of ψ_1 and ψ_2 to zero.

In the stationary and axisymmetric state the Euler potentials have the form of Eqs. (3.38). Comparing Eqs. (2.29) with Eqs. (3.38), we see that ψ_1 and ψ_2 behave as

$$\psi_1(t,r,\theta)\rightarrow\Psi_1(r,\theta),$$

$$\psi_2(t,r,\theta)\rightarrow-\Omega_F(\Psi_1)t+\Psi_2(r,\theta), \tag{3.59}$$

as the system approaches the stationary state. From Eq. (3.59), we further see that

$$(\dot{\psi}_1,\dot{\psi}_2)\rightarrow(0,-\Omega_F), \quad (\ddot{\psi}_1,\ddot{\psi}_2)\rightarrow(0,0), \tag{3.60}$$

respectively in the stationary limit. Then we can show that two components of Eq. (2.37) tend to Eqs. (3.56) and (3.57) substituting these limiting behaviors of $\dot{\psi}_i$ and $\ddot{\psi}_i$.

Similarly, the time derivatives of the canonical variables also do not vanish in the stationary and axisymmetric limit. From Eq. (2.41) and the first of Eq. (3.60), it turns out that (π_1,π_2) becomes

$$(\pi_1,\pi_2)\rightarrow\left(\frac{1}{4\pi}(\vec{\nabla}\Psi_1\cdot\vec{\nabla}\Psi_2)\Omega_F,-\frac{1}{4\pi}(\vec{\nabla}\Psi_1\cdot\vec{\nabla}\Psi_1)\Omega_F\right), \tag{3.61}$$

in the stationary limit. Differentiating Eq. (2.41) and also using Eqs. (3.60), we also find

$$(\dot{\pi}_1,\dot{\pi}_2)\rightarrow\left(-\frac{d\Omega_F}{d\Psi_1}\vec{\nabla}\Psi_1\cdot\vec{\nabla}\Psi_1\Omega_F,0\right). \tag{3.62}$$

Thus Eqs. (2.45) tend to equations

$$-\frac{d\Omega_F}{d\Psi_1}\Omega_F(\vec{\nabla}\Psi_1\cdot\vec{\nabla}\Psi_1)=\vec{\nabla}\cdot\left[-\Omega_F^2\vec{\nabla}\Psi_1+\frac{1}{r^2\sin^2\theta}\vec{\nabla}\Psi_1 + \vec{\nabla}\Psi_2\times(\vec{\nabla}\Psi_1\times\vec{\nabla}\Psi_2)\right] \tag{3.63}$$

and

$$0=-\vec{\nabla}\cdot[\vec{\nabla}\Psi_1\times(\vec{\nabla}\Psi_1\times\vec{\nabla}\Psi_2)], \tag{3.64}$$

where $\vec{\nabla}$ is an operator in the poloidal space. Therefore we can integrate the second of the above equations easily. Substituting the integral into the first equation, we arrive at the pulsar equation.

E. Scharlemann and Wagoner' s action principle

In the foregoing subsection, we have derived the equation of motion from the variational principle. However, there exists another action principle that leads to the pulsar equation. Namely, Scharlemann and Wagoner [3] showed that the pulsar equation (3.53) is derived from the action

$$I_{\rm SW}[\Psi_1]=-\frac{1}{8\pi}\int\left\{\frac{1}{\varpi^2}(1-\varpi^2\Omega_F^2)\times\left[(\partial_r\Psi_1)^2+\frac{1}{r^2}(\partial_\theta\Psi_1)^2\right]-\frac{1}{\varpi^2}B_T^2(\Psi_1)\right\}r^2\sin\theta dr d\theta, \tag{3.65}$$

where Ψ_1 is the field variable. In this subsection, we clarify the relation between these two action principles. Then it turns out that a force-free electromagnetic field with two invariant directions satisfying $F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)}=0$ generally has Scharlemann and Wagoner's type action. We show a method to construct Scharlemann and Wagoner's type action from the action of the force-free electromagnetic field given by Eq. (1.4).

1. Modified Lagrangian

Let us begin with an example in the particle dynamics. We consider a Lagrangian that has two degrees of freedom with one cyclic coordinate, i.e.,

$$L=L(q_1,\dot{q}_1,\dot{q}_2). \tag{3.66}$$

The Euler-Lagrange equation for q_2 yields an integral p as

$$\frac{\partial L}{\partial\dot{q}_2}(q_1,\dot{q}_1,\dot{q}_2)=p. \tag{3.67}$$

Solving this equation for $\dot{q}_2$, it is expressed as $\dot{q}_2=\chi(q_1,\dot{q}_1;p)$. Let us denote any function or equation in which $\dot{q}_2$ is replaced by $\chi(q_1,\dot{q}_1)$ by adding $\langle\ \rangle$. After eliminating $\dot{q}_2$, the equation of motion for q_1 becomes

$$\frac{d}{dt}\left\langle\frac{\partial L}{\partial\dot{q}_1}\right\rangle-\left\langle\frac{\partial L}{\partial q_1}\right\rangle=0. \tag{3.68}$$

The problem we are concerned with is to look for a Lagrangian that directly derives this equation as the Euler-Lagrange equation. By the relations

$$\frac{\partial\langle L\rangle}{\partial q_1}=\left\langle\frac{\partial L}{\partial q_1}\right\rangle+\left\langle\frac{\partial L}{\partial\dot{q}_2}\right\rangle\frac{\partial\chi}{\partial q_1}=\left\langle\frac{\partial L}{\partial q_1}\right\rangle+p\frac{\partial\chi}{\partial q_1} \tag{3.69}$$

and

$$\frac{\partial\langle L\rangle}{\partial\dot{q}_1}=\left\langle\frac{\partial L}{\partial\dot{q}_1}\right\rangle+\left\langle\frac{\partial L}{\partial\dot{q}_2}\right\rangle\frac{\partial\chi}{\partial\dot{q}_1}=\left\langle\frac{\partial L}{\partial\dot{q}_1}\right\rangle+p\frac{\partial\chi}{\partial\dot{q}_1}, \tag{3.70}$$

we find

$$\left\langle\frac{\partial L}{\partial\dot{q}_1}\right\rangle=\frac{\partial}{\partial\dot{q}_1}(\langle L\rangle-p\chi), \quad \left\langle\frac{\partial L}{\partial q_1}\right\rangle=\frac{\partial}{\partial q_1}(\langle L\rangle-p\chi). \tag{3.71}$$

Thus if we define a new Lagrangian $\tilde{L}$ as

$$\tilde{L}(q_1,\dot{q}_1)=\langle L\rangle - p\chi = \langle L\rangle - \left\langle \frac{\partial L}{\partial \dot{q}_1}\right\rangle \chi, \qquad (3.72)$$

the Euler-Lagrange equation derived from $\tilde{L}(q_2,\dot{q}_1)$ leads to Eq. (3.68). $\tilde{L}$ is called the modified Lagrangian [4] and $-\tilde{L}$ is called the Routhian [5].

2. Derivation of Scharlemann and Wagoner's action

Action (3.55) yields Eq. (3.56) as the Euler-Lagrange equation for Ψ_1. Substituting integral B_T into this equation, Ψ_2 is eliminated. Then we have the pulsar equation. This procedure is analogous to derivation of Eq. (3.68). We shall investigate this analogy in some detail.

Let us denote any function from which Ψ_2 is eliminated by means of $\langle\ \rangle$ again. Then the pulsar Eq. (3.53) is written as

$$\left\langle \frac{\partial' \mathcal{L}}{\partial \Psi_1} - \partial_i \frac{\partial' \mathcal{L}}{\partial(\partial_i \Psi_1)} \right\rangle = 0, \qquad (3.73)$$

where we abbreviate $\partial/\partial\Psi_1|_{\Psi_2=\mathrm{const}}$ as $\partial'/\partial\Psi_1$ and $\partial/\partial(\partial_i\Psi_1)|_{\Psi_2=\mathrm{const}}$ as $\partial'/\partial(\partial_i\Psi_1)$.

Integrating the Euler-Lagrange equation for Ψ_2, we have Eq. (3.54). However, this equation does not determine $\partial_i\Psi_2$ uniquely. Thus we cannot solve this for $\partial_i\Psi_2$. In fact, Eq. (3.54) determines Ψ_2 within the arbitrariness of Eq. (3.40). Namely, $\partial_i\Psi_2$ is not a function depending only on Ψ_1 and $\partial_i\Psi_1$. However, differentiating Eq. (3.50) by Ψ_1 and $\partial_i\Psi_1$, we have relations

$$\partial_r\Psi_1 \frac{\partial(\partial_\theta\Psi_2)}{\partial\Psi_1} - \frac{\partial(\partial_r\Psi_2)}{\partial\Psi_1}\partial_\theta\Psi_1 = \frac{1}{\sin\theta}\frac{dB_T}{d\Psi_1}, \qquad (3.74)$$

$$\partial_\theta\Psi_2 + \partial_r\Psi_1 \frac{\partial(\partial_\theta\Psi_2)}{\partial_r\Psi_1} - \frac{\partial(\partial_r\Psi_2)}{\partial_r\Psi_1}\partial_\theta\Psi_1 = 0, \qquad (3.75)$$

and

$$-\partial_r\Psi_2 + \partial_r\Psi_1 \frac{\partial(\partial_\theta\Psi_2)}{\partial_\theta\Psi_1} - \frac{\partial(\partial_r\Psi_2)}{\partial_\theta\Psi_1}\partial_\theta\Psi_1 = 0. \qquad (3.76)$$

Equation (3.54) holds irrespective of the arbitrariness in Ψ_2. Thus these relations are also satisfied independently of the arbitrariness in Ψ_2. Further, we can show that all the derivatives of $\partial_i\Psi_2$ appearing in the following are expressed by these forms. Thus $\partial_i\Psi_2$ can be treated as a function of Ψ_1 and $\partial_i\Psi_1$.

Using these relations, we have

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\Psi_1} &= \frac{\partial'\mathcal{L}}{\partial\Psi_1} + \frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\frac{\partial(\partial_i\Psi_2)}{\partial\Psi_1} \\ &= \frac{\partial'\mathcal{L}}{\partial\Psi_1} + \frac{\partial}{\partial\Psi_1}\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\partial_i\Psi_2\right) \\ &\quad -\left(\frac{\partial}{\partial\Psi_1}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right)\partial_i\Psi_2 \end{aligned} \qquad (3.77)$$

and

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_j\Psi_1)} &= \frac{\partial'\mathcal{L}}{\partial(\partial_j\Psi_1)} + \frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\frac{\partial(\partial_i\Psi_2)}{\partial(\partial_j\Psi_1)} \\ &= \frac{\partial'\mathcal{L}}{\partial(\partial_j\Psi_1)} + \frac{\partial}{\partial(\partial_j\Psi_1)}\left(\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\partial_i\Psi_2\right) \\ &\quad -\left(\frac{\partial}{\partial(\partial_j\Psi_1)}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right)\partial_i\Psi_2, \end{aligned} \qquad (3.78)$$

where $\partial/\partial\Psi_1$ and $\partial/\partial(\partial_i\Psi_1)$ denote the total derivatives in which the dependence of $\partial_i\Psi_2$ on Ψ_1 or $\partial_i\Psi_1$ is taken into account. Evidently, $\partial/\partial\Psi_1$ and $\partial/\partial(\partial_i\Psi_1)$ commute with $\langle\ \rangle$, but $\partial'/\partial\Psi_1$ and $\partial'/\partial(\partial_i\Psi_1)$ do not. Introducing $\bar{\mathcal{L}}$ as

$$\bar{\mathcal{L}} = \langle\mathcal{L}\rangle - \left\langle \frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\partial_i\Psi_2 \right\rangle, \qquad (3.79)$$

from Eqs. (3.77) and (3.78), we have

$$\begin{aligned}\left\langle \frac{\partial'\mathcal{L}}{\partial\Psi_1} - \partial_i\frac{\partial'\mathcal{L}}{\partial(\partial_i\Psi_1)}\right\rangle &= \frac{\partial\bar{\mathcal{L}}}{\partial\Psi_1} - \partial_j\frac{\partial\bar{\mathcal{L}}}{\partial(\partial_j\Psi_1)} \\ &\quad - \left\langle \partial_i\Psi_2 \frac{\partial}{\partial\Psi_1}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right\rangle \\ &\quad - \left\langle \partial_j\left(\partial_i\Psi_2\frac{\partial}{\partial(\partial_j\Psi_1)}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right)\right\rangle. \end{aligned} \qquad (3.80)$$

After some calculations, we find

$$\left\langle \partial_i\Psi_2\frac{\partial}{\partial\Psi_1}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right\rangle = -\frac{B_T}{4\pi\sin\theta}\frac{dB_T}{d\Psi_1} \qquad (3.81)$$

and

$$\left\langle \partial_j\left(\partial_i\Psi_2\frac{\partial}{\partial(\partial_j\Psi_1)}\frac{\partial\mathcal{L}}{\partial(\partial_i\Psi_2)}\right)\right\rangle = -\frac{B_T}{4\pi\sin\theta}\frac{dB_T}{d\Psi_1}. \qquad (3.82)$$

Thus the last two terms of Eq. (3.80) vanish. Accordingly, we can conclude that $\bar{\mathcal{L}}$ yields the pulsar equation as an Euler-Lagrange equation. In fact, we can see that $\bar{\mathcal{L}}$ indeed gives the Lagrangian density of Scharlemann and Wagoner's action.

Formal analogy between the modified action (3.72) and $\bar{\mathcal{L}}$ is evident. Thus Scharlemann and Wagoner's action is considered as the modified action of the action (3.55). Further, it is also evident that we can construct the modified action like Scharlemann and Wagoner's action in the configuration that has two invariant directions and satisfies $F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)}=0$.

F. Stationary and axisymmetric configuration with nonzero $F_{t\varphi}$

Now, we shall consider $F_{\mu\nu}\mathbf{k}^\mu\mathbf{m}^\nu = F_{t\varphi} \neq 0$ case. Integrating Eq. (3.33), we get the Euler potentials as

$$\phi_1 = t + \psi_1(r,\theta), \qquad \phi_2 = \kappa\varphi + \psi_2(r,\theta), \qquad (3.83)$$

where κ is a constant. The components of the electromagnetic field then become

$$F_{tr}=\partial_r\psi_2, \quad F_{t\theta}=\partial_\theta\psi_2, \quad F_{t\varphi}=\kappa, \quad F_{r\varphi}=\kappa\partial_r\psi_1,$$
$$F_{\theta\varphi}=\kappa\partial_\theta\psi_1, \quad F_{r\theta}=\partial_r\psi_1\partial_\theta\psi_2-\partial_r\psi_2\partial_\theta\psi_1. \tag{3.84}$$

This means that the φ component of the electric field $E_{(\varphi)}$, the component with respect to the orthonormal basis vector, is given by

$$E_{(\varphi)}=-(1/r\sin\theta)F_{t\varphi}=-\kappa/r\sin\theta. \tag{3.85}$$

Thus $E_{(\varphi)}$ becomes singular at the pole. Therefore, in the case of the stationary and axisymmetric configuration, the solution of $F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)}\neq 0$ type is unphysical as solutions containing the pole. However, this does not imply that solutions of this type are always unphysical. (The singularity in the stationary and axisymmetric configurations probably comes from the fact that the trajectory of Killing vector **m** is closed.) The purpose of this section is to present a method for handling the configuration with two symmetric directions. Since the existence of the singularity does not spoil this purpose, we complete the formulation.

Substituting the Euler potentials (3.83) into Lagrangian density (1.4), we have

$$\mathcal{L}=-\frac{r^2\sin\theta}{8\pi}\left\{\frac{\kappa}{r^2\sin^2\theta}\left[(\partial_r\psi_1)^2+\frac{1}{r^2}(\partial_\theta\psi_1)^2\right]\right.$$
$$+\frac{1}{r^2}(\partial_r\psi_1\partial_\theta\psi_2-\partial_r\psi_2\partial_\theta\psi_1)^2$$
$$\left.-\left[(\partial_r\psi_2)^2+\frac{1}{r^2}(\partial_\theta\psi_2)^2+\frac{\kappa^2}{r^2\sin^2\theta}\right]\right\}. \tag{3.86}$$

Then two Euler-Lagrange equations are given, respectively, by

$$\kappa^2\left[\partial_r\left(\frac{1}{\sin\theta}\partial_r\psi_1\right)+\frac{1}{r}\partial_\theta\left(\frac{1}{r\sin\theta}\partial_\theta\psi_1\right)\right]+\partial_r[\sin\theta(\partial_r\psi_1\partial_\theta\psi_2$$
$$-\partial_r\psi_2\partial_\theta\psi_1)]\partial_\theta\psi_2-\partial_\theta[\sin\theta(\partial_r\psi_1\partial_\theta\psi_2$$
$$-\partial_r\psi_2\partial_\theta\psi_1)]\partial_r\psi_2=0 \tag{3.87}$$

and

$$\partial_r(r^2\sin^2\theta\partial_r\psi_1)+\partial_\theta(\sin\theta\partial_\theta\psi_1)$$
$$+\partial_r[\sin\theta(\partial_r\psi_1\partial_\theta\psi_2-\partial_r\psi_2\partial_\theta\psi_1)]\partial_\theta\psi_1$$
$$-\partial_\theta[\sin\theta(\partial_r\psi_1\partial_\theta\psi_2-\partial_r\psi_2\partial_\theta\psi_1)]\partial_r\psi_1=0. \tag{3.88}$$

These are two independent components of the force-free equation. In contrast to the $F_{t\varphi}=0$ case, we can integrate neither of the two force-free equations immediately as Eq. (3.57). Thus we have to deal with two coupled equations for ψ_1 and ψ_2 simultaneously even under the stationary and axisymmetric condition. This is a general feature of $F_{\mu\nu}\zeta^\mu_{(1)}\zeta^\nu_{(2)}\neq 0$ type solutions.

As already mentioned, there does not exist a vector potential that yields the electromagnetic field (3.84) and also satisfies $\pounds_{\mathbf{k}}A_\mu=\pounds_{\mathbf{m}}A_\mu=0$. However, we can find an expression that satisfies either of $\pounds_{\mathbf{k}}A_\mu=0$ or $\pounds_{\mathbf{m}}A_\mu=0$. For example, a gauge transformation as $A_\mu\to\tilde{A}_\mu=A_\mu+\partial_\mu\lambda$ with

$$\lambda=-\frac{1}{2}\int\phi_2 dt=-\frac{1}{2}(\kappa\varphi+\psi_2)t \tag{3.89}$$

makes $\tilde{A}_\mu$ satisfy equation $\pounds_{\mathbf{k}}\tilde{A}_\mu=0$.

IV. SYMMETRY AND CONSERVED FLUX

A. Derivation of Noether's identities

There are close relations between the conserved quantities and the symmetry of the system. In our theory, the geometry of the conserved vector flux closely relates to the geometrical symmetry of the configuration. Namely, the symmetry of the system determines the surfaces along which the conserved vector fluxes flow. Since our basic equation is derived from the action principle, we can apply Noether's identities [6] systematically to this problem. Thus first we derive Noether's identities and express the conserved vector fluxes in the appropriate forms for this purpose.

The Lagrangian density of the present theory is a four-scalar density written as

$$\mathcal{L}=\mathcal{L}[\partial_\mu\phi_1,\partial_\mu\phi_2,g_{\mu\nu}]. \tag{4.1}$$

Lie differentiating this equation with respect to an arbitrary vector field ε^μ, we have

$$\partial_\mu(\mathcal{L}\varepsilon^\mu)=\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\pounds_\varepsilon(\partial_\mu\phi_1)+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\pounds_\varepsilon(\partial_\mu\phi_2)$$
$$+\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\pounds_\varepsilon g_{\mu\nu}. \tag{4.2}$$

Integrating by parts, we have

$$0=-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\right]\pounds_\varepsilon\phi_1-\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\right]\pounds_\varepsilon\phi_2$$
$$+\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\pounds_\varepsilon\phi_1+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\pounds_\varepsilon\phi_2-\mathcal{L}\varepsilon^\mu\right]$$
$$+\frac{\partial\mathcal{L}}{\partial g_{\mu\nu}}\pounds_\varepsilon g_{\mu\nu}. \tag{4.3}$$

We have introduced the metric energy-momentum tensor $T^{\mu\nu}$ by $T^{\mu\nu}=(2/\sqrt{-g})\partial\mathcal{L}/\partial g_{\mu\nu}$ in paper I. Further, we also define the canonical energy-momentum tensor by

$$\mathcal{T}^\mu_\nu=\frac{1}{\sqrt{-g}}\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\partial_\lambda\phi_1+\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\partial_\lambda\phi_2-\mathcal{L}\delta^\mu_\lambda\right]. \tag{4.4}$$

In the right-hand side of Eq. (4.3), the first and the second terms are the equation of motion (1.3). Thus they vanish if a solution of the basic equation is substituted for the Euler potentials. Under this condition, Eq. (4.3) is written as

$$(T^\mu_\lambda-\mathcal{T}^\mu_\lambda)\nabla_\mu\varepsilon^\lambda-(\nabla_\mu\mathcal{T}^\mu_\lambda)\varepsilon^\lambda=0. \tag{4.5}$$

In the above equation, ε_λ is an arbitrary vector field. Thus we can specify ε_λ and $\nabla_\mu \varepsilon_\lambda$ independently at any given point. Thus Eq. (4.5) implies

$$T^\mu_\lambda = \mathcal{T}^\mu_\lambda, \quad \nabla_\mu \mathcal{T}^\mu_\lambda = 0. \tag{4.6}$$

These simple relations are Noether's identities, arising from the general covariance of the theory. The metric energy-momentum tensor coincides with the canonical energy-momentum tensor in our theory. In other words, the spin density vanishes identically as expected from the fact that the basic variables are scalars. From Eqs. (4.6), further we have $\nabla_\mu T^\mu_\lambda = 0$.

Let ζ^μ denote a Killing vector field. Further, let $\Pi^\mu[\zeta]$ be the metric vector flux vector with respect to ζ and $P^\mu[\zeta]$ be the canonical vector flux with respect to ζ. Namely, $\Pi^\mu[\zeta]$ and $P^\mu[\zeta]$ are defined by

$$\Pi^\mu[\zeta] = T^\mu_\lambda \zeta^\lambda, \quad P^\mu[\zeta] = \mathcal{T}^\mu_\lambda \zeta^\lambda, \tag{4.7}$$

respectively. From the Killing equation, $\nabla_\mu \Pi^\mu[\zeta] = 0$ follows. Further, using the expression for $\mathcal{T}^\mu_\lambda$, we have

$$\Pi^\mu[\zeta] = P^\mu[\zeta] = \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \pounds_\zeta \phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \pounds_\zeta \phi_2 - \mathcal{L}\zeta^\mu\right]. \tag{4.8}$$

Although this equation seems trivial, the expression of the right-hand side gives a useful tool for studying the relation between the conserved vector fluxes and the symmetry of the system. This is because it is written by the Lie derivatives with respect to the Killing vector that directly reflect the symmetry of the system.

B. Properties of conserved vector flux

1. Stationary and axisymmetric configuration with $F_{t\varphi}=0$

The properties of the energy flux and the angular momentum flux in the stationary and axisymmetric configuration of $F_{t\varphi}=0$ type are already well known. Thus this configuration is appropriate to illustrate our method.

Let $\mathbf{k}^\mu$ be the stationary Killing vector and $\mathbf{m}^\mu$ be the axial Killing vector. Then the metric energy flux vector e^μ and the metric angular momentum flux vector l^μ are defined, respectively, by

$$e^\mu = -T^{\mu\nu}k_\nu = -\Pi^\mu[\mathbf{k}], \quad l^\mu = T^{\mu\nu}m_\nu = \Pi^\mu[\mathbf{m}]. \tag{4.9}$$

By Eq. (4.8), these fluxes are identical with the canonical ones. From the Euler potentials (3.38) and Eq. (4.8), we have

$$e^\mu = \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Omega_F + \mathcal{L}\mathbf{k}^\mu\right],$$

$$l^\mu = \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} - \mathcal{L}\mathbf{m}^\mu\right]. \tag{4.10}$$

Since both $\mathbf{k}^\mu$ and $\mathbf{m}^\mu$ have no poloidal components, the poloidal components of two vector fluxes relate as

$$e^A = \Omega_F l^A, \quad A = r, \theta. \tag{4.11}$$

Namely, the energy and the angular momentum flow along the same lines in the poloidal space. Further, making use of the explicit form of Lagrangian density (1.4), we find

$$\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \frac{1}{4\pi} F^{\mu\nu}\partial_\nu \phi_1 = \frac{1}{4\pi} F^{\mu A}\partial_A \Psi_1. \tag{4.12}$$

Thus we have

$$e^A \partial_A \Psi_1 = l^A \partial_A \Psi_1 = 0. \tag{4.13}$$

Thus the energy flux and the angular momentum flux flow along the poloidal field surface, i.e., the surface of constant Ψ_1 in the three-space of constant time. Relations (4.12) and (4.13) are well known in studies of the stationary and axisymmetric force-free magnetosphere. The present analysis shows that these are direct results of the symmetry and the form of the Lagrangian density.

2. Stationary and axisymmetric configuration with $F_{t\varphi}\neq 0$

We will consider the $F_{t\varphi}\neq 0$ type stationary and axisymmetric configuration. By the form of the Euler potentials (3.83), we have

$$e^\mu = -\frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} - \mathcal{L}\mathbf{k}^\mu\right],$$

$$l^\mu = \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \kappa - \mathcal{L}\mathbf{m}^\mu\right]. \tag{4.14}$$

Since

$$\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} = -\frac{1}{4\pi} F^{\mu\nu}\partial_\nu \phi_2,$$

$$\frac{1}{\sqrt{-g}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \frac{1}{4\pi} F^{\mu\nu}\partial_\nu \phi_1, \tag{4.15}$$

together with $\mathbf{k}^\mu \partial_\mu \phi_2 = 0$ and $\mathbf{m}^\mu \partial_\mu \phi_1 = 0$, we have

$$e^\mu \partial_\mu \phi_2 = 0, \quad l^\mu \partial_\mu \phi_1 = 0. \tag{4.16}$$

Thus the energy flux flows along the three-surface of constant ϕ_2. On the other hand, the angular momentum flux flow along the three-surface of constant ϕ_1.

3. Time-dependent axisymmetric configuration

In the case of the time-dependent axisymmetric configuration, we have

$$l^\mu = \frac{1}{\sqrt{-g}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} - \mathcal{L}\mathbf{m}^\mu\right], \tag{4.17}$$

from the Euler potentials (2.29). By means of Eq. (4.15) and $\mathbf{m}^{\mu}\partial_{\mu}\phi_1=0$, we find

$$l^{\mu}\partial_{\mu}\phi_1=0. \tag{4.18}$$

Therefore the angular momentum flux flows along the three-surface of constant ϕ_1. On the other hand, the energy flux does not have such a geometric property.

V. CONCLUDING REMARKS

We have illustrated a systematic method to deal with the force-free electromagnetic field with symmetry. Together with the theory given in paper I, our theory will extend applicability of the force-free approximation considerably. It allows us to study the magnetospheres under various conditions for symmetry from the unified point of view.

In this work we have studied the time-dependent axisymmetric field configuration as an example of the configuration with one direction of symmetry. We can apply this example directly to the evolution of the axisymmetric magnetosphere. Another example of the configuration with one direction of symmetry is the obliquely rotating pulsar magnetosphere. In this case, the electromagnetic field is invariant under the action generated by a linear combination of the stationary and axial Killing vector. The example of the configuration with two directions of symmetry treated here is the stationary and axisymmetric configuration. Probably another interesting application is the nonlinear plane wave. The plane wave is the simplest dynamical problem. In addition to this point, any wave tends to the plane wave in the region sufficiently away from the source. Thus this yields insight into the wave zone solution of the force-free electromagnetic field in the obliquely rotating pulsar magnetosphere. We plan to treat these applications elsewhere in future works.

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